Chapter - 3

Linear Diophantine Equation:

Def :- An equation of the form
$$ax+by=c$$
with $a\neq$, $b\neq 0$ and c integer, is called a linear diophantine equation in two unknowns x and y.

Theorem - 1

The linear diophantine equation $ax+by=c$ where a,b,c being integers has integer solutions if and only if d/c where $d =$ g.c.d of a and $d=(a,b)$

Moreover if $x=x_0, y=y_0$ is a particular solution, then any solution can be written as

$$x = x_0 + \frac{b}{d}\cdot t$$

$$y = y_0 - \frac{a}{d}t \quad \text{where } t \text{ is any integer.}$$

Proof :— Let us suppose that d/c then we have

$c = rd$, where r is any integer.

Now since $(a,b)=d$ then by definition, there exist integers x_1 and y_1 such that

$$ax_1+by_1 = d \quad \text{——— (1)}$$

multiplying both sides of (1) by $\frac{c}{d}$, we have

$$\frac{c}{d}ax_1 + \frac{c}{d}by_1 = d\cdot\frac{c}{d} = c$$

$$\Rightarrow c = a\left(\frac{c}{d}x_1\right) + b\left(\frac{c}{d}y_1\right) = ax+by$$

$\left(\frac{c}{d}x_1\right)$ and $\left(\frac{c}{d}\right)y_1$ satisfy the eqn (1)

Thus, linear diophantine equation has a solution.

Conversely, let us suppose that the equation

$ax+by=c$ has a solution, say (x_0, y_0)

Then $ax_0+by_0=c$

But ax_0+by_0 must be a multiple of d, i.e.

$ax_0+by_0=rd$ where r is any integer.

Therefore

$c=rd$

$\Rightarrow d/c$

Further, if $x=x_0$, $y=y_0$ is solution of (1)

then $ax_0+by_0=c$

Subtracting (1) from this equation, we get

$$a(x_0-x)+b(y_0-y)=0 \quad \text{(2)}$$

$$a(x_0-x)=b(y-y_0)$$

$$\Rightarrow a(x_0-x_1)=b(y_1-y_0) \quad \text{for } (x,y)=(x_1,y_1)$$

Now since $(a,b)=d$

There exist integers r_1 and r_2 such that

$a=r_1d$, $b=r_2d$

putting these values in (2) we get

$$r_1d[x_1-x_0]=-r_2d(y_1-y_0)$$

$$\Rightarrow r_1(x_1-x_0)=-r_2(y_1-y_0) \quad \text{(3)}$$

$$\Rightarrow \frac{x_1-x_0}{r_2}=-\frac{y_1-y_0}{r_1}=t$$

Therefore by division algorithm, we can write

$$y_1 = y_0 - t r_1$$

or $$y_1 = y_0 - \frac{a}{d} t$$

Now from ③ we have

$$r_1(x_1 - x_0) = r_2 r_1 t$$

$$\Rightarrow x_1 - x_0 = +r_2 t$$

$$\Rightarrow x_1 = x_0 + r_2 t$$

$$= x_0 + \frac{b}{d} t$$

Hence $x_1 = x_0 + \frac{b}{d} t$ and $y_1 = y_0 - \frac{a}{d} t$ is the general solution of ①

Example Determine if the linear diophantine equation

(i) $12x + 18y = 30$ (ii) $2x + 3y = 4$ (iii) ~~...~~

(iii) $6x + 8y = 25$ are solvable.

Solution: Comparing the given equations with $ax + by = c$ we have

(i) $a = 12,\ b = 18,\ c = 30$ and $(12, 18) = 6$ and $6/30$ so the linear diophantine equation $12x + 18y = 30$ has a solution.

(ii) Here $a = 2,\ b = 3,\ c = 4$ $(2, 3) = 1$ and $1/4$ so the linear diophantine equation $2x + 3y = 4$ also have a solution.

(iii) $a = 6,\ b = 8,\ c = 25$ and $(6, 8) = 2$ but $2 \nmid 25$ so the linear diophantine equation $6x + 8y = 25$ is not solvable.

Ex— Find the positive integer solution of (21)

$$7x + 19y = 213$$

Solution Dividing the given equation by the smallest coefficient 7 we get-

$$x = \frac{213 - 19y}{7} = 30 - 2y + \frac{3-5y}{7}$$

Since x is an integer, y is also a integer :

Therefore $\frac{3-5y}{7} = u$

is also a integer. Now we have

$$5y + 7u = 3$$

Dividing it by 5, we get

$$y = \frac{3-7u}{5}, = -u + \frac{3-2u}{5}$$

or $2u + 5v = 3$

Clearly $u = -1$, $v = 1$ is a solution. Hence $x = 25$, $y = 2$.

Thus the general solution of the given equation is

$$x = 25 + 19t$$
$$y = 2 - 7t$$

Since, we require the solution to be positive, i.e.

$$25 + 19t > 0, \quad 2 - 7t > 0$$

We require

$$-\frac{25}{19} < t < \frac{2}{7}$$

Thus $t = 0$ or $t = -1$. Hence the required positive integer solution are

$x = 25$, $y = 2$, $x = 6$, $y = 9$